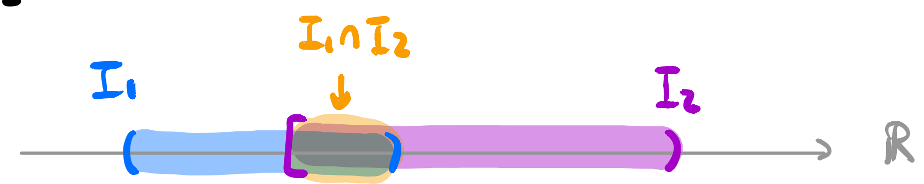
# MATH 2050 C Lecture 6 (Jan 27)

[ Problem Set 3 posted, due on Feb 11. ]

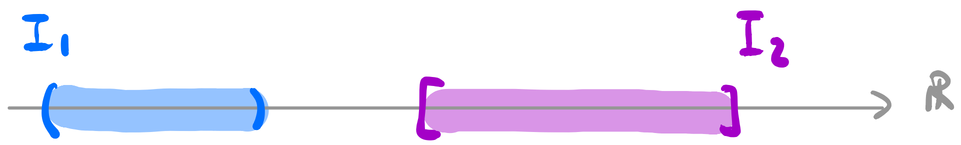
Last time ...... "connectedness" of intervals

Note: Given $I_1, I_2 \subseteq \mathbb{R}$ intervals.

- $I_1 \cap I_2$ is still an interval




- $I_1 \cup I_2$ may NOT be an interval




Q: What about $\bigcap_{i=1}^{\infty} I_i$ ?

Thm: ("Nested Interval Property" NIP)

Let $I_n := [a_n, b_n]$, $n \in \mathbb{N}$, be a sequence of closed and bounded intervals which are "nested":

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots\cdots$$

THEN: $\bigcap_{n=1}^{\infty} I_n \neq \phi$. Moreover, if

$\inf \{ \text{Length}(I_n) \mid n \in \mathbb{N} \} = 0$, then $\bigcap_{n=1}^{\infty} I_n = \{ \xi \}$.

**Examples:** $\displaystyle\bigcap_{n=1}^{\infty} [0, \tfrac{1}{n}] = \{0\}$

$$\bigcap_{n=1}^{\infty} [0, 1+\tfrac{1}{n}] = [0, 1] \neq \phi$$

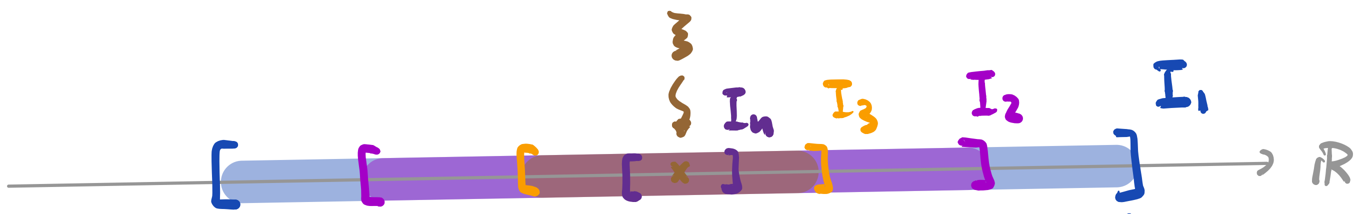
**Non-examples:**

(1) $\displaystyle\bigcap_{n=1}^{\infty} (0, \tfrac{1}{n}) = \phi$ not closed!

(2) $\displaystyle\bigcap_{n=1}^{\infty} [n, \infty) = \phi$ not bounded!

(3) $\displaystyle\bigcap_{n=1}^{\infty} [n, n+1] = \phi$ not nested!

**Proof of Thm:**




$$a_1 \leq a_2 \leq a_3 \cdots a_n \leq b_n \cdots b_3 \leq b_2 \leq b_1$$

Consider $\overset{\phi\neq}{S} = \{a_n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$, which is bdd above by $b_1$, so $\xi := \sup S \in \mathbb{R}$ exists by completeness of $\mathbb{R}$.

Claim: $\xi \in \bigcap_{n=1}^{\infty} I_n$ $\quad \left( \Rightarrow \bigcap_{n=1}^{\infty} I_n \neq \phi \right)$

Pf of Claim: Want $\xi \in I_n = [a_n, b_n]$ $\forall n \in \mathbb{N}$

- $\xi = \sup S$ is an upper bd of $S$

$$\Rightarrow \quad a_n \leq \xi \quad \forall n \in \mathbb{N}$$

- To see why $\boxed{\xi \leq b_n \quad \forall n \in \mathbb{N}}$, we argue by contradiction. Suppose NOT,

$$\exists\, m \in \mathbb{N} \text{ s.t. } \xi > b_m$$

Since $\xi = \sup S$ is the least upper bd.

$b_m$ cannot be an upper bd for $S$

$$\Rightarrow \exists\, k \in \mathbb{N} \text{ s.t. } b_m < a_k \in S$$

contradiction!

Case 1: $m < k \Rightarrow b_k \leq b_m < a_k \leq b_k$

Case 2: $m \geq k \Rightarrow a_m \leq b_m < a_k \leq a_m$

The remaining part is left as exercise.

$\blacksquare$

**Cor:** $\mathbb{R}$ is uncountable.

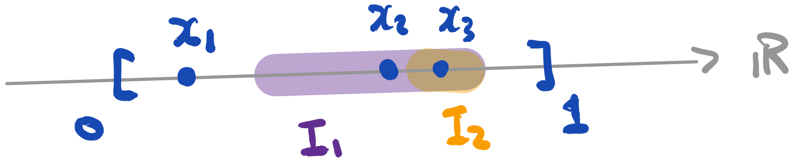
**Pf:** Suffices to show $[0,1]$ is uncountable.

Argue by contradiction.

Suppose $[0,1]$ is countable. Then, we can exhaust all the elements in $[0,1]$ in a sequence:

$$[0,1] = \{x_1, x_2, x_3, x_4, \ldots\ldots\} \quad \cdots\cdots (*)$$

We will construct a nested sequence of closed & bdd intervals $I_n$, $n \in \mathbb{N}$, inductively as follows:




- choose $I_1 \subseteq [0,1]$ s.t. $x_1 \notin I_1$
- choose $I_2 \subseteq I_1$ s.t. $x_2 \notin I_2$

$\vdots$ $\qquad\qquad\qquad \vdots$

- choose $I_n \subseteq I_{n-1}$ s.t. $x_n \notin I_n$

By NIP, $\bigcap_{n=1}^{\infty} I_n \neq \phi$, say $\xi \in \bigcap_{n=1}^{\infty} I_n \subseteq [0,1]$

contradiction!

By $(*)$, $\xi = x_k$ for some $k \in \mathbb{N}$ $\Rightarrow$ $\xi \notin I_k$

$\square$